

# Oblique discord

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Discord and entanglement characterize two kinds of quantum correlations, and discord captures more correlation than entanglement in the sense that even separable states may have nonzero discord. In this paper, we propose a new kind of quantum correlation we call it oblique discord. A zero-discord state corresponds to an orthonormal basis, while a zero-oblique-discord state corresponds to a basis which is not necessarily orthogonal. Under this definition, the set of zero-discord states is properly contained inside the set of zero-oblique-discord states, and the set of zero-oblique-discord states is properly contained inside the set of separable states. We give a characterization of zero-oblique-discord states via quantum operation, provide a geometric measure for oblique discord, and raise a conjecture with it holds we can define an information-theoretic measure for oblique discord. Also, we point out that, the definition of oblique discord can be properly extended to some different versions just as the case of quantum discord.

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## I. INTRODUCTION

Quantum correlation is one of the most striking features of quantum physics, and leads to powerful applications in quantum information science [1, 2]. Discord and entanglement characterize two kinds of quantum correlations, manifest complex structures and achieved fruitful results [1, 2]. Discord captures more correlation than entanglement in the sense that even separable states may have nonzero discord, although for certain measures discord not necessarily is larger than entanglement.

This paper asks the question: are there other kinds of correlation between entanglement and discord. To this aim, we properly generalize the definition of discord, we call the generalized version oblique discord. Under this definition, the set of zero-discord states is properly contained inside the set of zero-oblique-discord states, and the set of zero-oblique-discord states is properly contained inside the set of separable states. Moreover, we provide the information-theoretic measure and geometric measure for oblique discord compared to the case of discord, and propose the definition of global oblique discord compared to global discord.

This paper is organized as follows. In section 2, as preparations, we review the definitions of entanglement, discord, geometric discord, global discord and geometric global discord. In section 3, we provide the definition of oblique discord, give a characterization of zero-oblique-discord states via quantum operation, provide a geometric measure for oblique discord. Also, we raise a conjecture, if it holds we can define an information-theoretic measure for oblique discord. In section 4, we point out that, the definition of oblique discord can be properly extended to some different versions just as the case of quantum discord. In section 5, we give a summary.

## II. ENTANGLEMENT AND DISCORD

Suppose the quantum systems A, B are described by the complex Hilbert spaces  $H^A$  and  $H^B$ ,  $n_A = \dim H^A$  and  $n_B = \dim H^B$  are finite. The bipartite system AB is then described by the Hilbert space  $H^{AB} = H^A \otimes H^B$  with  $\dim H^{AB} = n_A n_B$ . Let  $I_A, I_B$  be the identity operators of A and B, then the identity operator of AB is  $I_{AB} = I_A \otimes I_B$ . When we consider an  $N$ -partite system  $A_1 A_2 \dots A_N$ , we use  $\{A_i\}_{i=1}^N$  to denote each subsystem and their Hilbert spaces are  $H^{A_i}$ , the dimension  $n_{A_i}$ , the identity  $I_{A_i}$ . We often omit the identity operator, for example we write  $\rho^A \otimes I_B$  as  $\rho^A$  by omitting  $I_B$ , without any confusion.

A quantum state  $\rho^{AB}$  is called a separable state if it can be written in the form

$$\rho^{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B, \quad (1)$$

where  $\sum_i p_i = 1, p_i \geq 0, \{\rho_i^A\}_i$  are states on  $H^A$  and  $\{\rho_i^B\}_i$  are states on  $H^B$ .  $\rho^{AB}$  is called an entangled state or disentangled state if it is not separable. By far many entanglement measures have been proposed [1].

A state  $\rho^{AB}$  is called a zero-discord state with respect to A if it can be written in the form

$$\rho^{AB} = \sum_{\alpha} p_{\alpha} |\alpha\rangle\langle\alpha| \otimes \rho_{\alpha}^B, \quad (2)$$

where  $\sum_{\alpha} p_{\alpha} = 1, p_{\alpha} \geq 0, \{|\alpha\rangle\}_{\alpha}$  is an orthonormal basis of  $H^A$ ,  $\{\rho_{\alpha}^B\}_{\alpha}$  are states on  $H^B$ .  $\rho^{AB}$  is called a discordant state if it is not a zero-discord state. The basic measure of discord is the information-theoretic measure proposed by [3, 4], that is

$$D^A(\rho^{AB}) = \inf_{\Pi_A} [I(\rho^{AB}) - I(\Pi_A \rho^{AB})], \quad (3)$$

where,  $I(\rho^{AB}) = S(\rho^A) + S(\rho^B) - S(\rho^{AB})$  is mutual information,  $\rho^A = \text{tr}_B \rho^{AB}$  is reduced state,  $S(\rho^{AB}) =$

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$-tr[\rho^{AB} \log_2 \rho^{AB}]$  is entropy function,  $\Pi_A$  denotes any projective measurement on A. It is shown [3] that

$$D^A(\rho^{AB}) \geq 0, \quad (4)$$

$$D^A(\rho^{AB}) = 0 \Leftrightarrow \rho^{AB} = \sum_{\alpha} p_{\alpha} |\alpha\rangle\langle\alpha| \otimes \rho_{\alpha}^B, \quad (5)$$

where  $\sum_{\alpha} p_{\alpha} = 1, p_{\alpha} \geq 0, \{|\alpha\rangle\}_{\alpha}$  is an orthonormal basis of  $H^A$ ,  $\{\rho_{\alpha}^B\}_{\alpha}$  are states on  $H^B$ . The intuitive meaning of  $D^A(\rho^{AB})$  is that it is the minimal loss of mutual information of  $\rho^{AB}$  over all projective measurement on A.

$D^A(\rho^{AB})$  is difficult to get the analytical expressions except for few special cases [5]. Another measure called geometric discord [6] is defined as

$$D_G^A(\rho^{AB}) = \inf\{tr[(\rho^{AB} - \chi^{AB})^2] : D^A(\chi^{AB}) = 0\}. \quad (6)$$

It is obvious that

$$D_G^A(\rho^{AB}) \geq 0, \quad (7)$$

$$D_G^A(\rho^{AB}) \Leftrightarrow D^A(\rho^{AB}) = 0. \quad (8)$$

For many cases,  $D_G^A(\rho^{AB})$  is easier to calculate than  $D^A(\rho^{AB})$  since  $D_G^A(\rho^{AB})$  avoids the complicated entropy function. For instance,  $D_G^A(\rho^{AB})$  allows analytical expressions for all  $2 \times d$  ( $2 \leq d < \infty$ ) states [6, 7].

Discord with respect to A,  $D^A(\rho^{AB})$ , can be extended to the definition of global discord [8] that (here we use the equivalent expression in [9])

$$D(\rho^{A_1 A_2 \dots A_N}) = \inf_{\Pi_{A_1} \Pi_{A_2} \dots \Pi_{A_N}} [I(\rho^{A_1 A_2 \dots A_N}) - I(\Pi_{A_1} \Pi_{A_2} \dots \Pi_{A_N} \rho^{A_1 A_2 \dots A_N})], \quad (9)$$

where  $I(\rho^{A_1 A_2 \dots A_N}) = \sum_{i=1}^N S(\rho^{A_i}) - S(\rho^{A_1 A_2 \dots A_N})$  is the mutual information of  $\rho^{A_1 A_2 \dots A_N}$ .  $D(\rho^{A_1 A_2 \dots A_N})$  has the property

$$\begin{aligned} D(\rho^{A_1 A_2 \dots A_N}) &= 0 \\ \Leftrightarrow \rho^{A_1 A_2 \dots A_N} &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \dots \sum_{i_N=1}^{n_N} p_{i_1 i_2 \dots i_N} |\alpha_{i_1}\rangle\langle\alpha_{i_1}| \\ &\quad \otimes |\alpha_{i_2}\rangle\langle\alpha_{i_2}| \otimes \dots \otimes |\alpha_{i_N}\rangle\langle\alpha_{i_N}|, \end{aligned} \quad (10)$$

where  $p_{i_1 i_2 \dots i_N} \geq 0, \sum_{i_1 i_2 \dots i_N} p_{i_1 i_2 \dots i_N} = 1, \{|\alpha_{i_j}\rangle\}_{i_j=1}^{n_j}$  is an orthonormal basis of  $H^{A_j}$ . For certain special states,  $D(\rho^{A_1 A_2 \dots A_N})$  possess analytical expressions [8, 9].

Geometric discord with respect to A,  $D_G^A(\rho^{AB})$ , can be extended to the definition of geometric global discord [10] that

$$D_G(\rho^{A_1 A_2 \dots A_N}) = \inf\{tr[(\rho^{A_1 A_2 \dots A_N} - \chi^{A_1 A_2 \dots A_N})^2] : D(\rho^{A_1 A_2 \dots A_N}) = 0\}. \quad (11)$$

For certain special states,  $D_G(\rho^{A_1 A_2 \dots A_N})$  possess analytical expressions [10].

The definitions of different kinds of discord above are all associated with the projective measurements. Projective measurements are a kind of most important quantum

operations, but not all quantum operations. There are many important problems, such as the optimal scheme to distinguish a set of quantum states, involve other quantum operations, rather than a projective measurement. A quantum operation is a map which maps a quantum state into another quantum state [11]. The familiar examples are projective measurement, general measurement, amplitude damping and phase damping of qubit, etc. In this paper, we relax the constraint of projective measurement, and seek a more general definition other than quantum discord.

### III. OBLIQUE DISCORD AND ITS MEASURES

**Definition 1.** We call the bipartite state  $\rho^{AB}$  a zero-oblique-discord state with respect to A, if  $\rho^{AB}$  can be written in the form

$$\rho^{AB} = \sum_{i=1}^{n_A} p_i |i\rangle\langle i| \otimes \rho_i^B, \quad (12)$$

where  $\sum_i p_i = 1, p_i \geq 0, \{|i\rangle\}_{i=1}^{n_A}$  is a normalized basis of  $H^A$ ,  $\{\rho_i^B\}_i$  are states on  $H^B$ . Notice that  $\{|i\rangle\}_{i=1}^{n_A}$  is not necessarily orthogonal.

Under this definition, combining Eqs.(1,2,12), we see that, the set of zero-discord states is properly contained inside the set of zero-oblique-discord, and the set of zero-oblique-discord states is properly contained inside the set of separable states, see Fig.1.

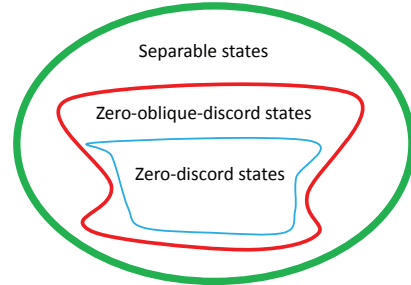


FIG. 1: Inclusion relations between the sets of separable states, zero-oblique-discord states and zero-discord states.

We give a characterization of zero-oblique-discord states via quantum operations. Suppose  $\{|i\rangle\}_{i=1}^{n_A}$  is a normalized basis of  $H^A$  which not necessarily orthogonal to each other. There exists a unique basis  $\{|\tilde{i}\rangle\}_{i=1}^{n_A}$  of  $H^A$  such that  $\langle i|\tilde{j}\rangle = \delta_{ij}$ , note that  $\{|\tilde{i}\rangle\}_{i=1}^{n_A}$  not necessarily orthogonal and not necessarily normalized.  $\{|\tilde{i}\rangle\}_{i=1}^{n_A}$  is called the dual basis of  $\{|i\rangle\}_{i=1}^{n_A}$ . We define the quantum operation  $\Phi_A = \{|i\rangle\langle\tilde{i}|\}_{i=1}^{n_A}$  which operates the bipartite state  $\rho^{AB}$  as

$$\Phi_A \rho^{AB} = \frac{\sum_{i=1}^{n_A} |i\rangle\langle\tilde{i}| \rho^{AB} |\tilde{i}\rangle\langle i|}{tr[\sum_{i=1}^{n_A} |\tilde{i}\rangle\langle\tilde{i}| \rho^{AB} |\tilde{i}\rangle\langle i|]}. \quad (13)$$

With this definition, we have Theorem 1 below.

**Theorem 1.** A bipartite state  $\rho^{AB}$  is a zero-oblique-discord state with respect to A, iff there exists an operation  $\Phi_A = \{|i\rangle\langle i|\}_{i=1}^{n_A}$  defined as in Eq.(13) such that

$$\Phi_A \rho^{AB} = \rho^{AB}. \quad (14)$$

**Proof.** Suppose there exists an operation  $\Phi_A = \{|i\rangle\langle i|\}_{i=1}^{n_A}$  such that  $\Phi_A \rho^{AB} = \rho^{AB}$ , we expand  $\rho^{AB}$  as

$$\rho^{AB} = \sum_{jk=1}^{n_A} \sum_{\lambda\mu=1}^{n_B} \rho_{jk,\lambda\mu} |j\rangle\langle k| \otimes |\lambda\rangle\langle\mu|, \quad (15)$$

where  $\{|j\rangle\}_{j=1}^{n_A} = \{|k\rangle\}_{k=1}^{n_A} = \{|i\rangle\}_{i=1}^{n_A}$ ,  $\{|\lambda\rangle\}_{\lambda=1}^{n_B} = \{|\mu\rangle\}_{\mu=1}^{n_B}$  is an orthonormal basis of  $H^B$ ,  $\rho_{jk,\lambda\mu} = \langle j|\lambda|\rho^{AB}|\tilde{k}\mu\rangle$ . Eq.(14) then reads

$$\rho^{AB} = \frac{\sum_{i=1}^{n_A} \sum_{\lambda\mu=1}^{n_B} \rho_{ii,\lambda\mu} |i\rangle\langle i| \otimes |\lambda\rangle\langle\mu|}{\sum_{i=1}^{n_A} \sum_{\lambda=1}^{n_B} \rho_{ii,\lambda\lambda}}, \quad (16)$$

it is of the form in Eq.(12).

Conversely, suppose  $\rho^{AB}$  can be expressed by Eq.(12), then  $\Phi_A = \{|i\rangle\langle i|\}_{i=1}^{n_A}$  fulfils  $\Phi_A \rho^{AB} = \rho^{AB}$ .  $\square$

Compared to the geometric measure of discord in Eq.(6), we propose the definition of geometric oblique discord as follows.

**Definition 2.** We define the geometric oblique discord of the bipartite state  $\rho^{AB}$  with respect to A as

$$D_{GO}^A(\rho^{AB}) = \inf_{\chi^{AB}} \{d(\rho^{AB}, \chi^{AB}) : \chi^{AB} \text{ is a zero-oblique-discord state}\}, \quad (17)$$

where,  $\inf$  runs over all zero-oblique-discord states  $\chi^{AB}$ ,  $d$  is a distance, for example,

$$d(\rho^{AB}, \chi^{AB}) = \text{tr}[(\rho^{AB} - \chi^{AB})^2]. \quad (18)$$

**Definition 3.** In the same spirit of Ref. [12], we can also define another geometric oblique discord as

$$D_{GO1}^A(\rho^{AB}) = \inf_{\Phi_A} \{d[\rho^{AB}, \Phi_A(\rho^{AB})] : \Phi_A \text{ is defined in Eq.(13)}\}, \quad (19)$$

once more for example,

$$d[\rho^{AB}, \Phi_A(\rho^{AB})] = \text{tr}\{[\rho^{AB} - \Phi_A(\rho^{AB})]^2\}. \quad (20)$$

**Definition 4.** Compared to the information-theoretic measure of discord in Eq.(3), it is very desirable to define an information-theoretic measure of oblique discord as

$$D_O^A(\rho^{AB}) = \inf_{\Phi_A} [I(\rho) - I(\Phi_A \rho)], \quad (21)$$

where  $I(\rho) = S(\rho^A) + S(\rho^B) - S(\rho)$  is the mutual information.

However, we do not know whether  $D_O^A(\rho^{AB})$  defined above is always nonnegative. Note that  $D_O^A(\rho^{AB}) \geq 0$  iff  $I(\rho^{AB}) \geq I(\Phi_A \rho^{AB})$  for any  $\Phi_A$ . We raise the conjecture below.

**Conjecture:**

$$I(\rho^{AB}) \geq I(\Phi_A \rho^{AB}) \text{ for any } \Phi_A \text{ and any } \rho^{AB}, \quad (22)$$

where  $I(\rho^{AB})$  is the mutual information,  $\Phi_A$  is defined in Eq.(13).

#### IV. EXTEND OBLIQUE DISCORD IN SOME WAYS

As in the case of discord, we can extend the definition of oblique discord in many ways.

**Definition 5.** An  $N$ -partite state  $\rho$  is said to be of zero global oblique discord, if it can be written in the form

$$\rho = \sum_{i_1=1}^{n_{A_1}} \sum_{i_2=1}^{n_{A_2}} \dots \sum_{i_N=1}^{n_{A_N}} p_{i_1 i_2 \dots i_N} |i_1\rangle\langle i_1| \otimes |i_2\rangle\langle i_2| \otimes \dots \otimes |i_N\rangle\langle i_N|, \quad (23)$$

where  $\sum_{i_1=1}^{n_{A_1}} \sum_{i_2=1}^{n_{A_2}} \dots \sum_{i_N=1}^{n_{A_N}} p_{i_1 i_2 \dots i_N} = 1$ ,  $p_{i_1 i_2 \dots i_N} \geq 0$ ,  $\{|i_j\rangle\}_{i_j=1}^{n_{A_j}}$  is a normalized basis of  $H^{A_j}$ . Notice that  $\{|i_j\rangle\}_{i_j=1}^{n_{A_j}}$  is not necessarily orthogonal.

**Theorem 2.** An  $N$ -partite state  $\rho$  is of zero global oblique discord iff there exists an operation  $\{\Phi_{A_j}\}_{j=1}^N = \{\{|i_j\rangle\langle i_j|\}_{i_j=1}^{n_{A_j}}\}_{j=1}^N$  such that

$$\Phi_{A_1 A_2 \dots A_N} \rho = \Phi_{A_1} \dots \Phi_{A_{N-1}} \Phi_{A_N} \rho = \rho. \quad (24)$$

It can be directly checked that

$$\Phi_{A_1}(\Phi_{A_2} \rho) = \Phi_{A_2}(\Phi_{A_1} \rho), \quad (25)$$

hence  $\Phi_{A_1 A_2 \dots A_N} \rho = \Phi_{A_1} \dots \Phi_{A_{N-1}} \Phi_{A_N} \rho$  above can be defined without any ambiguity.

**Definition 6.** We define the geometric global oblique discord of  $N$ -partite state  $\rho$  as

$$D_{GO}(\rho) = \inf_{\Phi_{A_1 A_2 \dots A_N}} d[\rho, \Phi_{A_1 A_2 \dots A_N}(\rho)], \quad (26)$$

where,  $d$  is a distance as in Eq.(7).

**Definition 7.** We define the global oblique discord of an  $N$ -partite state  $\rho$  as

$$D_O(\rho) = \inf_{\Phi_{A_1 A_2 \dots A_N}} [I(\rho) - I(\Phi_{A_1 A_2 \dots A_N} \rho)], \quad (27)$$

where  $I(\rho) = \sum_{j=1}^N S(\rho^{A_j}) - S(\rho)$  is the mutual information. Similar to the case of Eq.(21),  $D_O(\rho) \geq 0$  requires Eq.(22) holds.

#### V. SUMMARY AND DISCUSSION

The definition of quantum discord corresponds to orthogonal basis, in this paper, we relaxed the constraint of orthogonality, and proposed the definition of oblique discord. Oblique discord characterizes a new kind of quantum correlation between entanglement and discord.

There left many open questions for future investigations. Firstly, are there physical effects which can be revealed by oblique discord? Secondly, conjecture in Eq.(22) is true or false? Thirdly, how to calculate the different measures of oblique discord analytically or efficiently numerically, especially for  $n$ -qubit states.

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